Graph Theory
Homework 6

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September 28, 2018

Lemma 0.1 (for Exercise 1a). *Let $T_r(n)$ be the Turán graph and $t_r(n)$ the number of edges of $T_r(n)$. Then*

$$t_r(n+1) = t_r(n) + n - \left\lfloor \frac{n}{r} \right\rfloor$$

Proof. We form $t_r(n+1)$ from $t_r(n)$ by adding a vertex v to a group with $\lfloor \frac{n}{r} \rfloor$ vertices, then add all edges from v to vertices from other groups. There are n possible neighbors of v, but we must subtract the vertices from the same group. Thus we add $n - \lfloor \frac{n}{r} \rfloor$ edges. □

Note: The right inequality in the following proposition was not part of Exercise 1a, but I needed it for 1b, and it was more economical to prove both inequalities in the same proposition.

Proposition 0.2 (Exercise 1a). *Let $T_r(n)$ be the Turán graph and $t_r(n)$ the number of edges of $T_r(n)$. Then*

$$\left(1 - \frac{1}{r}\right)\binom{n}{2} \leq t_r(n) \leq \left(1 - \frac{1}{r}\right)\binom{n}{2} + n$$

Proof. We prove both inequalities by induction on n. The base case $n = 1$ holds because $\binom{1}{2} = 0$ and $t_r(1) = 0$. Assume the left inequality holds for $n = 1, \ldots, k$.

$$\begin{aligned} t_r(k+1) = t_r(k) + k - \left\lfloor \frac{k}{r} \right\rfloor &\geq \left(1 - \frac{1}{r}\right)\binom{k}{2} + \left(1 - \frac{1}{r}\right) k \\ = \left(1 - \frac{1}{r}\right)\left(\binom{k}{2} + k\right) &= \left(1 - \frac{1}{r}\right)\binom{k+1}{2} \end{aligned}$$

This completes the proof of the first inequality. For the second inequality, note that

$$k - \left\lfloor \frac{k}{r} \right\rfloor \leq k - \frac{k}{r} + 1$$

Then

$$\begin{aligned} t_r(k+1) &= t_r(k) + k - \left\lfloor \frac{k}{r} \right\rfloor \le \left(1 - \frac{1}{r}\right)\binom{k}{2} + k + k - \frac{k}{r} + 1 \\ &= \left(1 - \frac{1}{r}\right)\binom{k}{2} + \left(1 - \frac{1}{r}\right)k + (k+1) = \left(1 - \frac{1}{r}\right)\left(\binom{k}{2} + k\right) + (k+1) \\ &= \left(1 - \frac{1}{r}\right)\binom{k+1}{2} + (k+1) \end{aligned}$$

This completes the induction for the second inequality. □

Proposition 0.3 (Exercise 1b)**.** *Let $t_r(n)$ be as above. Then for a fixed $r \geq 1$,*

$$t_r(n) = \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 + o(n^2) \qquad \textit{as } n \to \infty$$

Proof. Using the first inequality from 1a,

$$t_r(n) - \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 \le \left(1 - \frac{1}{r}\right)\binom{n}{2} + n - \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 = \frac{1}{2}\left(1 - \frac{1}{r}\right)n$$

Using the second inequality from 1a,

$$\frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 - t_r(n) \le \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 - \left(1 - \frac{1}{r}\right)\binom{n}{2} = \frac{1}{2}\left(1 - \frac{1}{r}\right)n$$

Thus

$$\left| t_r(n) - \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 \right| \le \frac{1}{2}\left(1 - \frac{1}{r}\right)n$$

Thus for a fixed r, the error term is bounded above by a constant multiple of n. Thus for any $\epsilon > 0$, for sufficiently large n the error is bounded by ϵn^2. (Choose $n > \frac{c}{\epsilon}$ where $c = \frac{1}{2}\left(1 - \frac{1}{r}\right)$.) □

Proposition 0.4 (Exercise 2)**.** *The upper density of an infinite graph G lies in the set $\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\right\} = \left\{1 - \frac{1}{r} : r \in \mathbb{Z}_{\geq 1}\right\} \cup \{1\}$.*

Proof. Let $D(G)$ be the upper density of G and suppose $D(G) > 1 - \frac{1}{1-r}$ for some $r \geq 2$. Note that $D(G) - \left(1 - \frac{1}{1-r}\right) > 0$. Because $D(G)$ is the supremum over all densities of aribtrarily large finite subgraphs, for every $\delta > 0$ and $n_0 > 0$ there exists a finite subgraph $H_{\delta,n} \subset G$ with at least $n > n_0$ vertices and

$$D(H_{\delta,n}) > D(G) - \delta$$

Choose δ so that $0 < \delta < D(G) - \left(1 - \frac{1}{r-1}\right)$. Then choose ϵ with $0 < \epsilon < D(G) - \delta - \left(1 - \frac{1}{r-1}\right)$. Then

$$D(H_{\delta,n}) > D(G) - \delta > \left(1 - \frac{1}{r-1}\right) + \epsilon$$

We can write this inequality as

$$e(H_{\delta,n}) > \left(1 - \frac{1}{r-1} + \epsilon\right)\binom{n}{2} = \left(1 - \frac{1}{r-1} + \epsilon'\right)\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) > \left(1 - \frac{1}{r-1} + \epsilon\right)\frac{1}{2}n^2$$

choosing ϵ' so that $\frac{1}{2}n^2\epsilon' < \frac{1}{2}\epsilon n^2 - \frac{1}{2}\epsilon n$. Then by the Erdos-Stone Theorem, there exists n_0 so that $n > n_0$ implies $K_r(t_n) \subset H_{\delta,n}$ where $t_n \to \infty$ as $n \to \infty$. By the inequalities from Exercise 1a, the density of $K_r(t_n) = T_r(rt_n)$ tends toward $1 - \frac{1}{r}$ as $t_n \to \infty$, so $D(G) \geq 1 - \frac{1}{r}$. We have proven the implication

$$D(G) > 1 - \frac{1}{1-r} \implies D(G) \geq 1 - \frac{1}{r}$$

Thus it is impossible for $D(G)$ to lie in the interval $\left(1 - \frac{1}{1-r}, 1 - \frac{1}{r}\right)$ for any $r \geq 2$. Thus $D(G) \in \left\{1 - \frac{1}{r} : r \in \mathbb{Z}_{\geq 1}\right\} \cup \{1\}$. □

Theorem 0.5 (Exercise 3, Erdos-Simonovits Theorem)**.** *Let F be a graph with chromatic number $r = \chi(F)$. Then*

$$\text{ex}(F, n) = \frac{1}{2}\left(1 - \frac{1}{r-1}\right)n^2 + o(n^2)$$

Proof. Since $\chi(T_{r-1}(n)) = r - 1$, $T_{r-1}(n)$ does not contain F as a subgraph. Thus

$$\begin{aligned}\text{ex}(F, n) \geq e(T_{r-1n}(n)) = t_{r-1}(n) &\geq \left(1 - \frac{1}{r}\right)\binom{n}{2} \geq \left(1 - \frac{1}{r-1}\right)\binom{n}{2} \\ &= \left(1 - \frac{1}{r-1}\right)\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) \geq \frac{1}{2}\left(1 - \frac{1}{r-1}\right)n^2\end{aligned}$$

This is a sufficient lower bound for $\text{ex}(F, n)$. Now we obtain an upper bound. We can restate the definition of $\text{ex}(F, n)$ as

$$\text{ex}(F, n) \leq x \iff \Big(e(G) \geq x \implies F \subset G\Big) \tag{1}$$

Let $\epsilon > 0$. Then by Erdos-Stone, there exists n_0 such that for $n_0 \geq n$,

$$e(G) \geq \frac{1}{2}\left(1 - \frac{1}{r-1} + \epsilon\right)n^2 \implies K_r(t) \subset G$$

for some $t \geq \epsilon \log n/(2^{r+1}(r-1)!)$. Since $\chi(F) = r$, we know that $\chi(F) \subset K_r(t)$ for sufficiently large t, so $K_r(t) \subset G \implies F \subset G$. Then by our equivalence (1), for $n \geq n_0$, we have

$$\text{ex}(F, n) \leq \frac{1}{2}\left(1 - \frac{1}{r-1} + \epsilon\right)n^2 = \frac{1}{2}\left(1 - \frac{1}{r-1}\right)n^2 + \frac{1}{2}\epsilon n^2$$

Together, our two bounds say exactly that for $n \geq n_0$, we have

$$\text{ex}(F, n) = \frac{1}{2}\left(1 - \frac{1}{r-1}\right)n^2 + o(n^2)$$

□

(Exercise 4) A presentation of the quaternion group Q of order 8 is given by

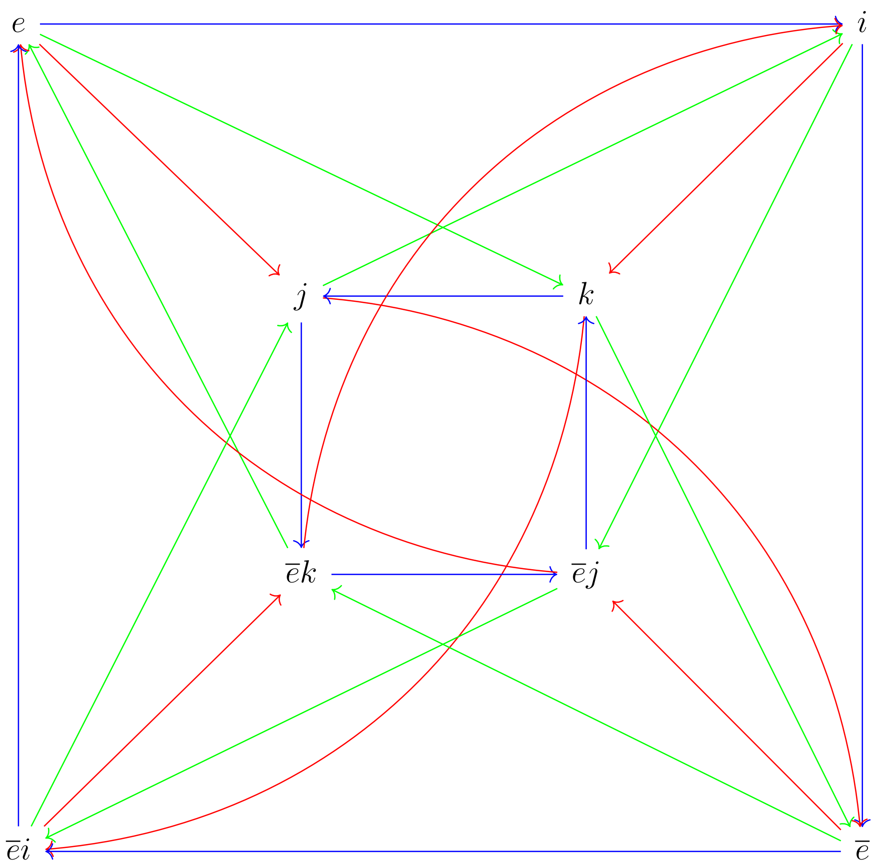
$$\langle \overline{e}, i, j, k | \overline{e}^2 = 1, i^2 = j^2 = k^2 = ijk = \overline{e} \rangle$$

It is clear from the relations that we do not need $\overline{e}$ or k as a generator. From the relations, we can deduce that $\overline{e}$ commutes with everything and that

$$ij = k \qquad ji = k \qquad ki = j \qquad ik = \overline{e}j \qquad jk = i \qquad kj = \overline{e}i$$

To show that this group has order 8, we draw the Cayley graph of this presentation. Because $\overline{e}$ is in the center, it is relatively clear how multiplication by $\overline{e}$ acts, so we omit the $\overline{e}$ arrows. We could also omit the k arrows, but we include them to better appreciate the symmetry.

Blue arrows are multiplication (on the right) by i, red arrows are multiplication (on the right) by j, and green arrows are multiplication (on the right) by k.



Proposition 0.6 (Exercise 5a). *Let A be a group with generators $\{g_i\}_{i \in I}$ and let $B \subset A$ be a subgroup. Then B is a normal subgroup of A if and only if for every vertex Ba in $G(A, B)$, there is a unique edge-label preserving graph automorphism $\phi : G(A, B) \to G(A, B)$ such that $\phi(B) = Ba$.*

Proof. Suppose B is normal in A, and let Ba be a vertex in $G(A, B)$. We have a map on vertices $\phi_a : G(A, B) \to G(A, B)$ which is $\phi_a(Bc) = aBc = Bac$. (These are equal because B is normal.) Clearly $\phi_a(B) = Ba$. Also, ϕ_a corresponds to left multiplication by a on A/B, so it is a bijection on vertices. It preserves the edge labelling, because because there is an edge $g_i : Bc \to Bc'$ if and only if $Bc' = Bcg_i$ if and only if there is an edge

$$\phi_a(Bc) = aBc \xrightarrow{g_i} (aBc)g_i = \phi_a(Bcg_i)$$

Thus ϕ_a is the required automorphism. Finally, we show uniqueness. Let ϕ, ϕ' both be edge-label preserving automorphisms of $G(A, B)$ with $\phi(B) = \phi'(B) = Ba$. We need to show that for arbitrary $c \in A$, we have $\phi(Bc) = \phi'(Bc)$. Write c as a product of generators $c = g_1 \dots g_n$. Then we have the following picture in $G(A, B)$.

$$B \xrightarrow{g_1} Bg_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} Bg_1 \dots g_n = Bc$$

Since ϕ, ϕ' are graph automorphisms, we also have

$$\phi(B) = Ba \xrightarrow{g_1} \phi(Bg_1) \xrightarrow{g_2} \dots \xrightarrow{g_n} \phi(Bc)$$

$$\phi'(B) = Ba \xrightarrow{g_1} \phi'(Bg_1) \xrightarrow{g_2} \dots \xrightarrow{g_n} \phi'(Bc)$$

Since ϕ, ϕ' are edge-label preserving, $\phi(Bg_1) = \phi'(Bg_1) = Bag_1$. Then continuing down the path with this reasoning, $\phi(Bc) = \phi'(Bc)$.

Now we suppose that $B \subset A$ is a subgroup so that $G(A, B)$ has this automorphism property. For each vertex Ba of $G(A, B)$, let ϕ_a be the corresponding automorphism. By the same sort of path-following uniqueness argument as above, $\phi_a(Bc) = Bac$ for any Bc. We also compute

$$\phi_{a_1}\phi_{a_2}(Bc) = \phi_{a_1}(Ba_2c) = Ba_1a_2c = \phi_{a_1a_2}(Bc) \implies \phi_{a_1}\phi_{a_2} = \phi_{a_1a_2}$$

Viewing A/B as a right coset space, we have an injective map of sets $\Phi : A/B \to \mathrm{Aut}(G(A, B))$, $Ba \mapsto \phi_a$. Φ is injective because if $Ba \neq Ba'$, then $\phi_a, \phi_{a'}$ take different values on B. We know that $B \subset A$ is normal if and only if the multiplication $(Ba_1)(Ba_2) = B(a_1a_2)$ is well defined, and by the previous equality,

$$\Phi(Ba_1a_2) = \phi_{a_1a_2} = \phi_{a_1}\phi_{a_2} = \Phi(Ba_1)\Phi(Ba_2)$$

Since Φ is injective, this says that $(Ba_1)(Ba_2) = B(a_1a_2)$ is well defined, so B is a normal subgroup. □

Proposition 0.7 (Exercise 5b). *Let A be a group with generators $\{a_i\}_{i \in I}$ and $B \subset A$ a subgroup. There can be no graph-theoretic criterion to be a necessary and sufficient condition for B to be a central subgroup.*

Proof. If A is any group with generators ranging over some indexing set I, then $G(A, A)$ is a single vertex with I loops. In particular, the Schreier graph $G(A, A)$ is the same (up to isomorphism) regardless of whether A is abelian or not.

If there was a criterion for central subgroups in terms of $G(A, B)$, then it would include the special case of $B = A$, where the question of being central is equivalent to A being abelian. Since $G(A, A)$ does not detect whether A is abelian, no such criterion exists. □